Backward blow-up estimates and initial trace for a parabolic system of reaction-diffusion

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Marie-Françoise BIDAUT-VERON* Marta GARCÍA-HUIDOBRO† Cecilia YARUR‡

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Abstract

In this article we study the positive solutions of the parabolic semilinear system of competitive type

$$\begin{cases} u_t - \Delta u + v^p = 0, \\ v_t - \Delta v + u^q = 0, \end{cases}$$

in $\Omega \times (0,T)$, where Ω is a domain of $\mathbb{R}^N$, and $p,q > 0$, $pq \neq 1$. Despite of the lack of comparison principles, we prove local upper estimates in the superlinear case $pq > 1$ of the form

$$u(x,t) \leqq Ct^{-(p+1)/(pq-1)}, \qquad v(x,t) \leqq Ct^{-(q+1)/(pq-1)}$$

in $\omega \times (0,T_1)$, for any domain $\omega \subset\subset \Omega$ and $T_1 \in (0,T)$, and $C = C(N,p,q,T_1,\omega)$. For $p,q > 1$, we prove the existence of an initial trace at time 0, which is a Borel measure on Ω. Finally we prove that the punctual singularities at time 0 are removable when $p,q \geqq 1 + 2/N$.

Keywords: Parabolic semilinear systems of reaction-diffusion, competitive systems, backward estimates, initial trace, singularities.

A.M.S. Classification: 35K45, 35K57, 35K58.

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1 Introduction

Let Ω be a domain of $\mathbb{R}^N$ $(N \geq 1)$ and $0 < T \leqq \infty$. In this work we are concerned with the positive solutions of the parabolic system with absorption terms

$$\left\{\begin{array}{l} u_t - \Delta u + v^p = 0, \\ v_t - \Delta v + u^q = 0, \end{array}\right. \tag{1.1}$$

in $\Omega \times (0, T)$, with $p, q > 0$, $pq \neq 1$, in particular in the superlinear case where $pq > 1$.

This system appears as a simple model of competition between two species, where the increase of the population of one of them reduces the growth rate of the other. Independently of the biological applications, it presents an evident interest, since it is the direct extension of the scalar equation

$$U_t - \Delta U + U^Q = 0, \tag{1.2}$$

with $Q \neq 1$. For $Q > 1$, any nonnegative subsolution of equation (1.2) in $\Omega \times (0, T)$ satisfies the following upper estimate: for any bounded C^2 domain $\omega \subset \Omega$

$$U(x,t) \leqq ((Q-1)t)^{-1/(Q-1)} + Cd(x, \partial\omega)^{-2/(Q-1)} \qquad \forall (x,t) \in \omega \times (0,T), \tag{1.3}$$

where $d(x, \partial\omega)$ is the distance from x to the boundary of ω and $C = C(N, Q)$, see [15]. This estimate follows from the comparison principle, as shown at Proposition 3.4. Moreover it was proved in [15] that any solution U of equation (1.2) in $\Omega \times (0, T)$ admits a trace at time 0 in the following sense:

There exist two disjoints sets $\mathcal{R}$ and $\mathcal{S}$ such that $\mathcal{R} \cup \mathcal{S} = \Omega$, and $\mathcal{R}$ is open, and a nonnegative Radon measure μ on $\mathcal{R}$, such that

- For any $x_0 \in \mathcal{R}$, and any $\psi \in C_c^0(\mathcal{R})$,

$$\lim_{t \to 0} \int_{\mathcal{R}} U(.,t)\psi = \int_{\mathcal{R}} \psi d\mu,$$

- For any open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{S} \neq \emptyset$,

$$\lim_{t \to 0} \int_{\mathcal{U}} u(.,t) = \infty.$$

Moreover the trace $(\mathcal{S}, \mu)$ is unique whenever $Q < 1 + 2/N$.

Up to now, system (1.1) has been barely touched on. Indeed an essential difficulty appears: *the lack of results for comparison principles.* As a consequence, most of the classical properties of equation (1.2), based on the use of standard supersolutions, are hardly extendable. Some existence results are given in [12] for bounded initial data, and then in [3] for more general multipower systems with non smooth data, see also [13] for quasilinear operators. Otherwise the existence of traveling waves is treated in [9]. For the associated elliptic system

$$\left\{\begin{array}{l} -\Delta u + v^p = 0, \\ -\Delta v + u^q = 0, \end{array}\right. \tag{1.4}$$

the isolated singularities are completely described in [4] for the superlinear case $pq > 1$ and for the sublinear case $pq < 1$, see also [17], [18] for $p, q \geqq 1$. The study shows a great complexity of the possible singularities; in particular many nonradial singular solutions are constructed by bifurcation methods. The question of large solutions of system (1.4) is studied in the radial case in [11], showing unexpected multiplicity results, and the behavior of the solutions near the boundary is open in dimension $N > 1$; the existence is an open problem in the general case. For such competitive problems, some more adapted sub-supersolutions and super-subsolutions have been introduced, see [14], [16], [3], [10], but the problem remains to construct them. The uniqueness is also a difficult problem, as it was first observed in [1].

Our first result consists in local backward upper estimates for the solutions of the system: defining the two exponents

$$a = \frac{p+1}{pq-1}, \qquad b = \frac{q+1}{pq-1}, \tag{1.5}$$

we obtain the following:

Theorem 1.1 *Assume that* $pq > 1$. *Let* (u, v) *be a positive solution of system (1.1) in* $\Omega \times (0, T)$. *Then for any domain* $\omega \subset\subset \Omega$ ($\omega = \mathbb{R}^N$ *if* $\Omega = \mathbb{R}^N$),

$$u(x,t) \leqq Ct^{-a}, \qquad v(x,t) \leqq Ct^{-b}, \qquad \forall (x,t) \in \omega \times (0,T), \tag{1.6}$$

for some $C = C(N, p, q, T, \omega)$.

Our second result is the existence of a trace in the following sense:

Theorem 1.2 *Assume that* $p, q > 1$. *Let* (u, v) *be a positive solution of the system in* $\Omega \times (0, T)$. *Then there exist two disjoints sets* $\mathcal{R}$ *and* $\mathcal{S}$ *such that* $\mathcal{R} \cup \mathcal{S} = \Omega$, *and* $\mathcal{R}$ *is open, and nonnegative Radon measures* μ_1, μ_2 *on* $\mathcal{R}$, *such that the following holds:*

- *For any* $x_0 \in \mathcal{R}$, *and any* $\psi \in C_c^0(\mathcal{R})$,

$$\lim_{t \to 0} \int_{\mathcal{R}} u(.,t)\psi = \int_{\mathcal{R}} \psi d\mu_1, \qquad \lim_{t \to 0} \int_{\mathcal{R}} v(.,t)\psi = \int_{\mathcal{R}} \psi d\mu_2. \tag{1.7}$$

- *For any open set* $\mathcal{U}$ *such that* $\mathcal{U} \cap \mathcal{S} \neq \emptyset$,

$$\lim_{t \to 0} \int_{\mathcal{U}} (u(.,t) + v(.,t)) = \infty. \tag{1.8}$$

As a consequence we can define a notion of trace of (u, v) at time 0:

Definition 1.3 *The couple $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ of Borel measures $\mathcal{B}_1$,$\mathcal{B}_2$ on Ω associated to the triplet $(\mathcal{S}, \mu_1, \mu_2)$ defined for $i = 1$,2 by*

$$\mathcal{B}_i(E) = \begin{cases} \mu_i(E) & \textit{if } E \subset \mathcal{R}, \\ \infty & \textit{if } E \cap \mathcal{S} \neq \emptyset, \end{cases}$$

is called the initial trace of (u, v).

Finally we give a result of removability of the initial singularities inspired by [6, Theorem 2]:

Theorem 1.4 *Assume that $p, q \geqq 1+2/N$. If there exists a positive solution (u, v) of system (1.1) in $\Omega \times (0, T)$ such that*

$$\lim_{t \to 0} \int_\Omega (u(.,t) + v(t))\varphi = 0, \qquad \forall \varphi \in C_c^\infty (\Omega \backslash \{0\}), \tag{1.9}$$

then $u, v \in C^{2,1}(\Omega \times [0, T))$ and $u(x, 0) = v(x, 0) = 0$, $\forall x \in \Omega$.

In each section we point out some questions which remain open.

2 Some existence results

Next we recall some results that we obtained in [3] where we studied the existence and the eventual uniqueness of signed solutions of the Cauchy problem with initial data (u_0, v_0)

$$\begin{cases} u_t - \Delta u + |v|^p |u|^{-1} u = 0, \\ v_t - \Delta v + |u|^q |v|^{-1} v = 0, \end{cases} \tag{2.1}$$

where $p, q > 0$, and

$$|u|^{-1} u = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0. \end{cases}$$

In particular we showed in [3] the following results:

Theorem 2.1 *Assume that Ω is bounded. Suppose that $u_0 \in L^\theta(\Omega)$ and $v_0 \in L^\lambda(\Omega)$ with $1 \leqq \theta, \lambda \leqq \infty$, with*

$$\max(\frac{p}{\lambda}, \frac{q}{\theta}) < 1 + 2/N,$$

or that u_0,v_0 are two bounded Radon measures in Ω, and

$$\max(p, q) < 1 + 2/N. \tag{2.2}$$

Then there exists a weak solution (u, v) of the system with Dirichlet or Neuman conditions on the lateral boundary, such that for any $\psi \in C_c^0(\Omega)$,

$$\lim_{t \to 0} \int_{\mathcal{R}} u(.,t)\psi = \int_{\mathcal{R}} \psi du_0, \qquad \lim_{t \to 0} \int_{\mathcal{R}} v(.,t)\psi = \int_{\mathcal{R}} \psi dv_0. \tag{2.3}$$

Also, there exist two solutions (u_1, v_1) and (u_2, v_2) such that any solution (u, v) satisfies $u_1 \leqq u \leqq u_2$ and $v_2 \leqq v \leqq v_1$.

Moreover, if $p, q \geq 1$ and $u_0 \in L^\theta(\Omega)$ and $v_0 \in L^\lambda(\Omega)$ with

$$\max(\frac{p}{\lambda} - \frac{1}{\theta}, \frac{q}{\theta} - \frac{1}{\lambda}) < \frac{2}{N}, \tag{2.4}$$

then (u, v) is unique; in particular this holds for any $u_0, v_0 \in L^1(\Omega)$, if (2.2) is satisfied, or if $u_0, v_0 \in L^\theta(\Omega)$ with $\theta \geq N(\max(p, q) - 1)/2$.

3 Local a priori estimates

When looking for local upper estimates of the nonnegative solutions of system (2.1) near $t = 0$, we notice that the system admits the solution $(0, v)$ with v a solution of the heat equation in $\Omega \times (0, T)$, for which we have no estimate, since the set of solutions is a vector space. That is why we suppose that u and v are *positive* in $\Omega \times (0, T)$. The question of upper estimates for one of the functions is very closely linked to the question of lower estimates for the other one.

We define a solution of problem (1.1) in $\Omega \times (0, T)$ as a couple (u, v) of positive functions such that $u \in L^q_{loc}(\Omega \times (0, T))$, $v \in L^p_{loc}(\Omega \times (0, T))$ and

$$\iint_{\Omega \times (0,T)} (-u\varphi_t - u\Delta\varphi + v^p\varphi) = 0, \tag{3.1}$$

$$\iint_{\Omega \times (0,T)} (-v\varphi_t - v\Delta\varphi + u^q\varphi) = 0, \tag{3.2}$$

for any $\varphi \in \mathcal{D}(\Omega \times (0, T))$. From the standard regularity theory for the heat equation it follows that $u, v \in C^{2,1}_{loc}(\Omega \times (0, T))$, and then $u, v \in C^\infty(\Omega \times (0, T))$ since u, v are positive.

As in the case of the scalar equation (1.2), the system (1.1) admits a particular solution (u^*, v^*) for $pq > 1$, defined by

$$u^*(t) = A^* t^{-a}, \qquad v^*(t) = B^* t^{-b},$$

where

$$(A^*)^{pq-1} = (p + 1)(q + 1)^p(pq - 1)^{-(p+1)}, \qquad (B^*)^{pq-1} = (q + 1)(p + 1)^q(pq - 1)^{-(q+1)}.$$

In [4], the authors studied the singularities near 0 of the positive solutions of the associated elliptic system (1.4) in $B(0, 1)\backslash \{0\}$. System (1.4) admits particular solutions when $\min(2a, 2b) > N - 2$, given by

$$u_*(x) = A_* |x|^{-2a}, \qquad v_*(x) = B_* |x|^{-2b},$$

with

$$A_*^{pq-1} = 2a(2a + 2 - N)((2b(2b + 2 - N))^p, \qquad B_*^{pq-1} = 2b(2b + 2 - N)((2a(2a + 2 - N))^p.$$

When $pq > 1$ the following upper estimates hold near near 0 :

$$u(x) \leqq C\,|x|^{-2a}, \qquad v(x) \leqq C\,|x|^{-2b},$$

for some $C = C(p, q, N)$. The proofs were based on estimates of the mean value of u and v on the sphere $\{|x| = r\}$, on the mean value inequality for subharmonic functions, and a bootstrap technique for comparisons between different spheres.

For system (1.1) the estimates (1.6) are based on local integral estimates of the solutions, following some ideas of [5] for elliptic systems with source terms. Then we use two arguments: the mean value inequality in suitable cylinders for subsolutions of the heat equation, and an adaptation of the bootstrap technique of [4].

Notation 3.1 *For any cylinder $\tilde{Q} = \omega \times (s, t) \subset \Omega \times (0, T)$ and any $w \in L^1(\tilde{Q})$ we set*

$$⨏\!\!\!\!\int_{\tilde{Q}} w = \frac{1}{\left|\tilde{Q}\right|} \int_s^t \int_\omega w.$$

For any $\rho > 0$, we define the open ball $B_\rho = B(0, \rho)$ and the cylinder

$$\tilde{Q}_\rho = B_\rho \times \left[-\rho^2, 0\right].$$

We denote by ξ_1 the first eigenfunction of the Laplacian in B_1, such that $\int_{B_1} \xi_1 = 1$, with eigenvalue λ_1, and by ξ the first eigenfunction in B_ρ with eigenvalue $\lambda_{1,\rho} = \lambda_1/\rho^2$, defined by

$$\xi(x) = \xi_1(\frac{x}{\rho}), \qquad \forall x \in B_\rho. \tag{3.3}$$

First we need a precise version of the mean value inequality.

Lemma 3.2 *Let Ω be any domain in $\mathbb{R}^N$, and let w be a subsolution of the heat equation in $\Omega \times (0, T)$, with $w \in C^{2,1}(\Omega \times (0, T))$. Then for any $r > 0$, there exists a constant $C = C(N, r)$, such that for any (x_0, t_0) and $\rho > 0$ such that $(x_0, t_0) + \tilde{Q}_\rho \subset \Omega \times (0, T)$, and for any $\varepsilon \in (0, 1/2)$,*

$$\sup_{(x_0,t_0)+\tilde{Q}_{\rho(1-\varepsilon)}} w \leqq C\varepsilon^{-\frac{N+2}{r^2}} \left(⨏\!\!\!\!\int_{(x_0,t_0)+\tilde{Q}_\rho} w^r \right)^{\frac{1}{r}}. \tag{3.4}$$

Proof. This Lemma is given in case $\varepsilon = 1$ in [8] for solutions of the heat equation, and we adapt its proof with the parameter ε. We can assume that $(x_0, t_0) = 0$ and $r \in (0, 1)$ .From [8] there exists $C_N = C(N) > 0$ such that for any $\sigma \in (0, 1)$,

$$\sup_{\tilde{Q}_{\rho\sigma}} w \leqq C_N (1 - \sigma)^{-(N+2)} ⨏\!\!\!\!\int_{\tilde{Q}_\rho} w. \tag{3.5}$$

For any $n \in \mathbb{N}$, let $\rho_n = \rho(1-\varepsilon)(1+\varepsilon/2+...+(\varepsilon/2)^n)$, and $M_n = \sup_{\tilde{Q}_{\rho_n}} |w|$. From (3.5) we obtain

$$M_n \leqq C_N (1 - \frac{\rho_n}{\rho_{n+1}})^{-(N+2)} \fint\!\!\!\fint_{\tilde{Q}_{\rho_{n+1}}} w;$$

thus with a new constant C_N

$$M_n \leqq C_N \varepsilon^{-(n+1)(N+2)} \fint\!\!\!\fint_{\tilde{Q}_{\rho_{n+1}}} w.$$

From Young inequality, for any $\delta \in (0,1)$, we obtain

$$\begin{aligned} M_n &\leqq C_N \varepsilon^{-(n+1)(N+2)} M_{n+1}^{1-r} \fint\!\!\!\fint_{\tilde{Q}_{\rho_{n+1}}} w^r \\ &\leqq \delta M_{n+1} + r\delta^{1-1/r} (C_N \varepsilon^{-(n+1)(N+2)})^{\frac{1}{r}} \left(\fint\!\!\!\fint_{\tilde{Q}_{\rho_{n+1}}} w^r \right)^{\frac{1}{r}} \end{aligned}$$

Defining $D = r\delta^{1-1/r} C_N{}^{\frac{1}{r}}$ and $b = \varepsilon^{-(N+2)/r}$, we find

$$M_n \leqq \delta M_{n+1} + b^{n+1} D \left(\fint\!\!\!\fint_{\tilde{Q}_{\rho_{n+1}}} w^r \right)^{\frac{1}{r}}.$$

Taking $\delta = 1/2b$ and iterating, we obtain

$$\begin{aligned} M_0 = \sup_{\tilde{Q}_{\rho(1-\varepsilon)}} |w| &\leqq \delta^{n+1} M_{n+1} + bD \sum_{i=0}^{n} (\delta b)^i \left(\fint\!\!\!\fint_{\tilde{Q}_{\rho_{n+1}}} w^r \right)^{\frac{1}{r}} \\ &\leqq \delta^{n+1} M_{n+1} + 2bD \left(\fint\!\!\!\fint_{\tilde{Q}_{\rho_{n+1}}} w^r \right)^{\frac{1}{r}}. \end{aligned}$$

Since $\tilde{Q}_{\rho_{n+1}} \subset \tilde{Q}_{\rho(1+\varepsilon)}$, we deduce (3.4) by going to the limit as $n \to \infty$. ■

Next we recall a bootstrap result given from [4, Lemma 2.2]:

Lemma 3.3 *Let $d, h, \ell \in \mathbb{R}$ with $d \in (0,1)$ and y, Φ be two continuous positive functions on some interval $(0, R]$. Assume that there exist some $C, M > 0$ and $\varepsilon_0 \in (0, 1/2]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$,*

$$y(r) \leqq C\, \varepsilon^{-h} \Phi(r)\, y^d \left[r(1-\varepsilon) \right] \qquad \text{and} \quad \max_{\tau \in [r/2, r]} \Phi(\tau) \leqq M\, \Phi(r),$$

or else

$$y(r) \leqq C\, \varepsilon^{-h} \Phi(r)\, y^d \left[r(1+\varepsilon) \right] \qquad \text{and} \quad \max_{\tau \in [r, 3r/2]} \Phi(\tau) \leqq M\, \Phi(r),$$

for any $r \in (0, R/2]$. Then there exists another $C > 0$ such that

$$y(r) \leqq C\, \Phi(r)^{1/(1-d)}$$

on $(0, R/2]$.

Next we prove the estimates (1.6).

Proof of Theorem 1.1. We consider any point $(x_0, t_0) \in \Omega \times (0, T)$, and any $\rho > 0$ such that $B(x_0, \rho) = x_0 + B_\rho \subset \Omega$. By translation we can reduce to the case $x_0 = 0$. For given $s \in (0, 1)$, we consider a smooth function $\eta_0(t)$ on $[-2s, 0]$ with values in $[0, 1]$ such that $\eta_0 = 1$ in $[-s, 0]$ and $\eta_0(-2s) = 0$ and $0 \leqq (\eta_0)_t(t) \leqq Cs^{-1}$. Choosing s such that $0 < t_0 - 2s < t_0$, we set $\eta(t) = \eta_0(t - t_0)$. We multiply the first equation in (1.1) by

$$\varphi = \xi^\lambda(x)\eta^\lambda(t),$$

where ξ is defined at (3.3), and $\lambda > 1$, which will be chosen large enough. We obtain

$$\frac{d}{dt}\left(\int_{B_\rho} u\xi^\lambda\eta^\lambda(t)\right) + \int_{B_\rho} v^p\xi^\lambda\eta^\lambda = \lambda\int_{B_\rho} u\xi^\lambda\eta^{\lambda-1}\eta_t(t) + \int_{B_\rho} u(\Delta\xi^\lambda)\eta^\lambda. \tag{3.6}$$

By computation, we find

$$\rho^2\Delta\xi^\lambda(x) = \Delta\xi_1^\lambda(\frac{x}{\rho}) = -\lambda\lambda_1\xi_1^\lambda(\frac{x}{\rho}) + \lambda(\lambda-1)\xi_1^{\lambda-2}\left|\nabla\xi_1\right|^2(\frac{x}{\rho}).$$

For given $\ell > 1$, if $\lambda > 2\ell'$, the function $g_\ell(y) = \xi_1^{\lambda/\ell'-2}\left|\nabla\xi_1\right|^2$ is bounded, thus

$$\begin{aligned}
\int_{B_\rho} u(.,t)(\Delta\xi^\lambda)\eta^\lambda(t) &\leqq \frac{\lambda(\lambda-1)}{\rho^2}\int_{B_\rho} u(x,t)(\xi_1^{\lambda-2}\left|\nabla\xi_1\right|^2)(\frac{x}{\rho})\eta^\lambda(t)dx \\
&= \frac{\lambda(\lambda-1)}{\rho^2}\int_{B_\rho} u(x,t)\xi^{\lambda/\ell}g_\ell(\frac{x}{\rho})\eta^\lambda(t)dx \\
&\leqq \frac{\lambda(\lambda-1)}{\rho^2}\left(\int_{B_\rho} u(.,t)^\ell\xi^\lambda\eta^\lambda(t)\right)^{1/\ell}\left(\int_{B_\rho} g_\ell^{\ell'}(\frac{x}{\rho})\eta^\lambda(t)dx\right)^{1/\ell'} \\
&\leqq C\rho^{N/\ell'-2}\left(\int_{B_\rho} u(.,t)^\ell\xi^\lambda\eta^\lambda(t)\right)^{1/\ell}
\end{aligned}$$

and even with different constants $C = C(N, \ell)$

$$\begin{aligned}
\int_{B_\rho} u(.,t)\left|\Delta\xi^\lambda\right|\eta^\lambda(t) &\leqq \lambda\lambda_1\rho^{-2}\int_{B_\rho} u(.,t)\xi^\lambda\eta^\lambda(t) + C\rho^{N/\ell'-2}\left(\int_{B_\rho} u(.,t)^\ell\xi^\lambda\eta^\lambda(t)\right)^{1/\ell} \\
&\leqq C\rho^{N/\ell'-2}\left(\int_{B_\rho} u(.,t)^\ell\xi^\lambda\eta^\lambda(t)\right)^{1/\ell}.
\end{aligned} \tag{3.7}$$

Moreover

$$\begin{aligned}
&\int_{B_\rho} u(.,t)\xi^\lambda\eta^{\lambda-1}\eta_t(t) \leqq Cs^{-1}\left(\int_{B_\rho} u(.,t)^\ell\xi^\lambda\eta^\lambda(t)\right)^{1/\ell}\left(\int_{B_\rho} \xi^\lambda\eta^{\lambda-\ell'}(t)\right)^{1/\ell'} \\
&\leqq C\rho^{N/\ell'}s^{-1}\left(\int_{B_\rho} u(.,t)^\ell\xi^\lambda\eta^\lambda(t)\right)^{1/\ell}.
\end{aligned}$$

Integrating (3.6) on (t_0-2s,t_0) , and using Hölder inequality,

$$\int_{B_\rho} u(.,t_0)\xi^\lambda + \int_{t_0-2s}^{t_0}\int_{B_\rho} v^p\xi^\lambda\eta^\lambda \leqq C\rho^{N/\ell'}(\rho^{-2}+s^{-1})\int_{t_0-2s}^{t_0}\left(\int_{B_\rho} u^\ell\xi^\lambda\eta^\lambda\right)^{1/\ell}$$
$$\leqq C\rho^{N/\ell'}(\rho^{-2}+s^{-1})s^{1/\ell'}\left(\int_{t_0-2s}^{t_0}\int_{B_\rho} u^\ell\xi^\lambda\eta^\lambda\right)^{1/\ell}. \quad (3.8)$$

In the same way, for any $\kappa>1$, if $\lambda>2k'$,

$$\int_{B_\rho} v(.,t_0)\xi^\lambda + \int_{t_0-2s}^{t_0}\int_{B_\rho} u^q\xi^\lambda\eta^\lambda \leqq C\rho^{N/\kappa'}(\rho^{-2}+s^{-1})s^{1/\kappa'}(\int_{t_0-2s}^{t_0}\int_{B_\rho} v^\kappa\xi^\lambda\eta^\lambda)^{1/\kappa}. \quad (3.9)$$

Next we discuss according to the values of p and q.

First case: $p,q>1$. We take $\ell=q,\kappa=p$, and $2s=\rho^2$ and consider any t_0 such that $0<t_0-\rho^2<t_0<T$. Let us denote $Q_\rho=(0,t_0)+\tilde{Q}_\rho$. Then

$$\iint_{Q_\rho} v^p\xi^\lambda\eta^\lambda \leqq C\rho^{(N+2)/q'-2}\left(\iint_{Q_\rho} u^q\xi^\lambda\eta^\lambda\right)^{1/q},$$
$$\iint_{Q_\rho} u^q\xi^\lambda\eta^\lambda \leqq C\rho^{(N+2)/p'-2}\left(\iint_{Q_\rho} v^p\xi^\lambda\eta^\lambda\right)^{1/p},$$

that means

$$\fint\!\!\fint_{Q_\rho} v^p\xi^\lambda\eta^\lambda \leqq C\rho^{-2}\left(\fint\!\!\fint_{Q_\rho} u^q\xi^\lambda\eta^\lambda\right)^{1/q,} \quad (3.10)$$
$$\fint\!\!\fint_{Q_\rho} u^q\xi^\lambda\eta^\lambda \leqq C\rho^{-2}\left(\fint\!\!\fint_{Q_\rho} v^p\xi^\lambda\eta^\lambda\right)^{1/p}.$$

Hence

$$\fint\!\!\fint_{Q_\rho} u^q\xi^\lambda\eta^\lambda \leqq C\rho^{-2(p+1)/p}\left(\fint\!\!\fint_{Q_\rho} u^q\xi^\lambda\eta^\lambda\right)^{1/pq}.$$

Then we get an estimate of the form

$$(\fint\!\!\fint_{Q_{\rho/2}} u^q)^{1/q} \leqq \frac{C}{\rho^{2(p+1)/(pq-1)}} \quad (3.11)$$

and similarly

$$(\fint\!\!\fint_{Q_{\rho/2}} v^p)^{1/p} \leqq \frac{C}{\rho^{2(q+1)/(pq-1)}} \quad (3.12)$$

But u is a subsolution of the heat equation, hence there exists a $C=C(N,q)$ such that

$$u(x,t) \leqq C(\fint\!\!\fint_{Q_{\rho/2}} u^q)^{1/q},$$

from Lemma 3.2 with $r = q$ and $\varepsilon = 1$. Taking $\rho^2 = t_0/2M$, with $M > 1$, we deduce the estimates

$$u(x,t) \leqq \frac{C}{t^{(p+1)/(pq-1)}}, \qquad v(x,t) \leqq \frac{C}{t^{(q+1)/(pq-1)}},$$

for any $t \in (0,T)$ and any $x \in \Omega$ such that $B(x, \sqrt{t/2M}) \subset \Omega$, with $C = C(N,p,q,M)$.Then (1.6) follows.

General case: $pq > 1$. We can assume $p \leqq 1 < q$. Taking again $0 < t_0 - \rho^2 < t_0 < T$ and $2s = \rho^2$, and using (3.8) with $\ell = q > 1$, we find again (3.10). Using (3.9), we find for any $\kappa > 1$,

$$\iint_{Q_\rho} u^q \xi^\lambda \eta^\lambda \leqq C\rho^{(N+2)/\kappa'-2} \left(\iint_{Q_\rho} v^\kappa \xi^\lambda \eta^\lambda \right)^{1/\kappa} \leqq C\rho^{(N+2)/\kappa'-2} \sup_{Q_\rho} v^{1-p/\kappa} \left(\iint_{Q_\rho} v^p \right)^{1/\kappa}. \tag{3.13}$$

More precisely, for any $\varepsilon \in (0,1/2)$, from Lemma 3.2, we find taking $r = p$ and $\kappa = q$,

$$\sup_{Q_\rho} v \leqq C\varepsilon^{-(N+2)/p^2} \rho^{-(N+2)/p} \left(\iint_{Q_{\rho(1+\varepsilon)}} v^p \right)^{1/p},$$

then

$$\begin{aligned} \sup_{Q_\rho} v^{1-p/q} \left(\iint_{Q_\rho} v^p \right)^{1/q} &\leqq C\varepsilon^{-(N+2)\frac{(q-p)}{p^2 q}} \rho^{-(N+2)\frac{(q-p)}{pq}} \left(\iint_{Q_{\rho(1+\varepsilon)}} v^p \right)^{\frac{(q-p)}{pq}+\frac{1}{q}} \\ &= C\varepsilon^{-(N+2)\frac{(q-p)}{p^2 q}} \rho^{-(N+2)\frac{(q-p)}{pq}} \left(\iint_{Q_{\rho(1+\varepsilon)}} v^p \right)^{1/p}. \end{aligned}$$

Using (3.13) we deduce

$$\begin{aligned} \iint_{Q_{\rho(1-\varepsilon)}} u^q &\leqq C\varepsilon^{-(2\lambda+(N+2)\frac{(q-p)}{p^2 q})} \rho^{(N+2)/q'-2-(N+2)\frac{(q-p)}{pq}} \left(\iint_{Q_{\rho(1+\varepsilon)}} v^p \right)^{1/p} \\ &= C\varepsilon^{-(2\lambda+(N+2)\frac{(q-p)}{p^2 q})} \rho^{(N+2)/(1-1/p)-2} \left(\iint_{Q_{\rho(1+\varepsilon)}} v^p \right)^{1/p}; \end{aligned}$$

setting $h = 2\lambda + (N+2)(q-p)/p^2 q$, that means

$$⨌_{Q_{\rho(1-\varepsilon)}} u^q \leqq C\varepsilon^{-h} \rho^{-2} \left(⨌_{Q_{\rho(1+\varepsilon)}} v^p \right)^{1/p}. \tag{3.14}$$

Next from (3.10) we have

$$⨌_{Q_{\rho(1-\varepsilon)}} v^p \leqq C\rho^{-2} \left(⨌_{Q_{\rho(1+\varepsilon)}} u^q \right)^{1/q}, \tag{3.15}$$

thus changing $\rho(1-\varepsilon)$ into $\rho(1+\varepsilon)$,

$$\iint_{Q_{\rho(1+\varepsilon)}} v^p \leqq C\rho^{-2} \left(\iint_{Q_{\rho(1+4\varepsilon)}} u^q \right)^{1/q}.$$

Hence from (3.14), we deduce

$$\iint_{Q_{\rho(1-\varepsilon)}} u^q \leqq C\varepsilon^{-h} \rho^{-2(p+1)/p} \left(\iint_{Q_{\rho(1+4\varepsilon)}} u^q \right)^{1/pq}.$$

From Lemma 3.3, we conclude that

$$\left(\iint_{Q_\rho} u^q \right)^{(pq-1)/q} \leqq C\rho^{-2(p+1)}.$$

Hence (3.11) follows as above, and then (3.12) from (3.15), and the conclusion follows again.

■

Next we give a first extension of the scalar estimate (1.3) to system (1.1), using some ideas of [4, p. 243].

Proposition 3.4 *Let $q \geqq p > 1$. Let (u, v) be any positive solution of system (1.1) in $\Omega \times (0, T)$, where Ω is a bounded C^2 domain Then there exists a constant $C = C(N, p, q)$ such that*

$$u^{(q+1)/(p+1)}(x, t) + v(x, t) \leqq C(t + d^2(x, \partial\Omega))^{-1/(p-1)}, \qquad \forall (x, t) \in \Omega \times (0, T) \qquad (3.16)$$

Proof. Let $F = (k + u)^d + v$, with $d = (q + 1)/(p + 1) > 1$ and $k > 0$. Then

$$\begin{aligned} F_t - \Delta F &= d(k + u)^{d-1}(u_t - \Delta u) - d(d - 1)(k + u)^{d-2} |\nabla u|^2 + v_t - \Delta v \\ &\leqq -d(k + u)^{d-1} v^p - u^q. \end{aligned}$$

But $(k + u)^q \leqq 2^{q-1}(k^q + u^q)$, thus

$$F_t - \Delta F + d(k + u)^{d-1} v^p + 2^{1-q}(k + u)^q \leqq k^q.$$

Observe that $(k + u)^q = (k + u)^{d-1}(k + u)^{dp}$, and $F^p \leqq 2^{p-1}((k + u)^{dp} + v^p)$. Then

$$F_t - \Delta F + c(k + u)^{d-1} F^p \leqq k^q,$$

with $c = 2^{1-p} \min(d, 2^{1-q})$. In particular, taking $k = c^{-1/(d-1)}$, F is a subsolution of equation

$$U_t - \Delta U + U^p = K \qquad (3.17)$$

in $\Omega \times (0, T)$, where $K = k^q = K(p, q)$. Let $f(t) = ((p - 1)t))^{-1/(Q-1)}$ and let g be the maximal solution of the stationary problem $-\Delta U + U^p = 0$ in Ω such that $g = \infty$ on $\partial\Omega$.

Then for any $\varepsilon > 0$, the function $(x,t) \mapsto G_\varepsilon(x,t) = K^{1/p} + f(t-\varepsilon) + g(x)$ is a supersolution of equation (3.17) in $\Omega \times (\varepsilon, T)$. Going to the limit as $\varepsilon \longrightarrow 0$, it follows that

$$F(x,t) \leqq K^{1/p} + f(t) + g(x)$$

in $\Omega \times (0,T)$; then there exists a constants $C' = C'(N,p)$ such that

$$F(x,t) \leqq K^{1/p} + f(t) + C' d(x, \partial\Omega)^{-2/(p-1)}, \qquad \forall (x,t) \in \Omega \times (0,T),$$

and the conclusion follows. ∎

Open problem: The estimate (3.16) does not appear to be optimal, except in the case $p = q$ where $u = v$ is a solution of the scalar equation (1.2). Can we obtain for $p, q > 1$, and even for $pq > 1$, an upper estimate in $\Omega \times (0,T)$ of the form

$$u(x,t) \leqq C(t + d^2(x,\partial\Omega))^{-a}, \qquad v(x,t) \leqq C(t + d^2(x,\partial\Omega))^{-b},$$

with $C = C(N,p,q)$?

4 Initial trace

First we show some properties available for any $p, q > 0$.

Lemma 4.1 *Assume $p, q > 0$. Let (u,v) be any positive solution of system (1.1), and let $B(x_0,\rho) \subset \Omega$. If $\int_0^T \int_{B(x_0,\rho)} v^p < \infty$, then $\int_{B(x_0,\bar{\rho})} u(.,t)$ is bounded as $t \to 0$ for any $\bar{\rho} < \rho$, and there exists a Radon measure $m_{1,\rho}$ on $B(x_0,\rho)$ such that for any $\psi \in C_c^\infty(B(x_0,\rho))$,*

$$\lim_{t \to 0} \int_{B(x_0,\rho)} u(.,t)\psi = m_{1,\rho}(\psi).$$

Proof. We reduce to the case $x_0 = 0$. We set

$$X(t) = \int_{B_\rho} u(.,t)\xi^\lambda, \quad Y(t) = \int_{B_\rho} v(.,t)\xi^\lambda, \quad Z(t) = \int_{B_\rho} u^q(.,t)\xi^\lambda, \quad W(t) = \int_{B_\rho} v^p(.,t)\xi^\lambda. \tag{4.1}$$

where ξ is defined at (3.3) and $\lambda \geqq 2$. We obtain

$$\begin{aligned} X_t + W &= \frac{d}{dt}\left(\int_{B_\rho} u\xi^\lambda\right) + \int_{B_\rho} v^p \xi^\lambda = \int_{B_\rho} u(\Delta \xi^\lambda) \\ &= -\lambda\lambda_{1,\rho} \int_{B_\rho} u\xi^\lambda + \lambda(\lambda-1) \int_{B_\rho} u\xi^{\lambda-2} |\nabla\xi|^2 \\ &\geqq -\lambda\lambda_{1,\rho} \int_{B_\rho} u\xi^\lambda = -\lambda\lambda_{1,\rho} X, \end{aligned}$$

hence

$$\frac{d}{dt}(e^{\lambda\lambda_{1,\rho}t} X(t)) + e^{\lambda\lambda_{1,\rho}t} W(t) \geqq 0.$$

By integration we obtain for any $t < \theta$

$$e^{\lambda\lambda_{1,\rho}\theta}X(\theta) - e^{\lambda\lambda_{1,\rho}t}X(t) + \int_t^\theta e^{\lambda\lambda_{1,\rho}s}W(s)ds \geqq 0;$$

and from our assumption $W \in L^1((0,T))$. Then $e^{\lambda\lambda_{1,\rho}t}X(t)$ is bounded, and in turn $X(t)$ is bounded. Then $\int_{B_\rho} u(.,t)\xi^\lambda$ is bounded, hence $\int_{B(x_0,\bar{\rho})} u(.,t)$ is bounded. Let $\psi \in C_c^\infty(B(x_0,\rho))$. Then

$$\frac{d}{dt}\int_{B_\rho} u(.,t)\psi + \int_{B_\rho} v^p\psi = \int_{B_\rho} u(\Delta\psi).$$

Since $\Delta\psi$ is bounded with compact support, we have $|\psi| + |\Delta\psi| \leqq C\xi^\lambda$ for some positive constant C, and thus $\int_{B_\rho} u(\Delta\psi)$ is bounded, implying $\int_{B_\rho} u(.,t)\psi$ has a limit $m_{1,\rho}(\psi)$, which defines a Radon measure $m_{1,\rho}$ on B_ρ. ■

Lemma 4.2 *Assume $p, q > 0$. Let (u, v) be any positive solution of system (1.1), and let $B(x_0, \rho_0) \subset \Omega$. If $\int_{B(x_0,\rho_0)} u(.,t)$ is bounded as $t \to 0$, then*

(i) for any $\rho < \rho_0$, $\int_t^\theta \int_{B_\rho} v^p$ is bounded;

(ii) for any $\bar{\rho} < \rho_0$, any $1 \leqq \sigma < 1 + 2/N$, and any $0 < t < \theta < T$

$$\int_t^\theta \int_{B_{\bar{\rho}}} u^\sigma dx \leqq C, \tag{4.2}$$

where $C = C(N, p, q, \bar{\rho}, \rho_0, \sigma)$.

Proof. We still reduce to the case $x_0 = 0$.

(i) Let $0 < t < \theta < T$ with fixed θ, and $C = \sup_{(0,\theta]} \int_{B_{\rho_0}} u(.,t)$. Let $\psi \in C_c^\infty(B_{\rho_0})$ with values in $[0,1]$ such that $\psi = 1$ on B_ρ. Taking ψ as a test function in the equation in u and integrating between t and θ, we find

$$\frac{d}{dt}\left(\int_{B_{\rho_0}} u\psi\right) + \int_{B_{\rho_0}} v^p\psi = \int_{B_{\rho_0}} u(\Delta\psi) \leqq C\left\|\Delta\psi\right\|_{L^\infty(\Omega)},$$

thus

$$\int_{B_{\rho_0}} u(.,\theta)\psi + \int_t^\theta \int_{B_{\rho_0}} v^p\psi \leqq C(\|\Delta\psi\|_{L^\infty(\Omega)} + 1),$$

hence $\int_t^\theta \int_{B_\rho} v^p$ is bounded.

(ii) Here we use the ideas of [2, Propositions 2.1,2.2.] relative to quasilinear equations in order to estimate the gradient. Since $\sigma < 1 + 2/N$, we can fix $\alpha = \alpha(\sigma)$ such that

$$-1 < \alpha < 0 \text{ and } \sigma \leqq \alpha + 1 + 2/N. \tag{4.3}$$

Let ρ be fixed such that $\bar{\rho} < \rho < \rho_0$. We multiply the equation in u by $(1+u)^\alpha \xi^\lambda$, where ξ is defined at (3.3), with $\lambda \geqq 2/|\alpha|$. Then we find for fixed $\theta < T$, and any $0 < t \leqq \theta$

$$\begin{aligned} &\frac{1}{\alpha+1}\int_{B_\rho}(1+u(.,t))^{\alpha+1}\xi^\lambda + |\alpha| \int_t^\theta \int_{B_\rho}(1+u)^{\alpha-1}|\nabla u|^2 \xi^\lambda \\ &= \frac{1}{\alpha+1}\int_{B_\rho}(1+u(.,\theta))^{\alpha+1}\xi^\lambda + \int_t^\theta \int_{B_\rho} v^p(1+u)^\alpha \xi^\lambda + \lambda \int_t^\theta \int_{B_\rho}(1+u)^\alpha \xi^{\lambda-1}\nabla u.\nabla\xi. \end{aligned}$$

Applying twice the Hölder inequality, we find

$$\begin{aligned} &\frac{1}{\alpha+1}\int_{B_\rho}(1+u(.,t))^{\alpha+1}\xi^\lambda + \frac{1}{2}|\alpha| \int_t^\theta \int_{B_\rho}(1+u)^{\alpha-1}|\nabla u|^2 \xi^\lambda \\ &\leqq C + \int_t^\theta \int_{B_\rho} v^p(1+u)^\alpha \xi^\lambda + C\int_t^\theta \int_{B_\rho}(1+u)^{\alpha+1}\xi^{\lambda-2}|\nabla\xi|^2 \\ &\leqq C + \int_t^\theta \int_{B_\rho} v^p + C\left(\int_t^\theta \int_{B_\rho}(1+u)\xi^\lambda\right)^{1+\alpha}\left(\int_t^\theta \int_{B_\rho}\xi^{\lambda-2/|\alpha|}|\nabla\xi|^{2/|\alpha|}\right)^{|\alpha|}, \end{aligned} \tag{4.4}$$

where C depends on θ and σ. Since $\int_{B_\rho} u(.,t)\xi^\lambda$ is bounded, and $\int_t^\theta \int_{B_\rho} v^p$ is bounded, we obtain an estimate of the gradient:

$$\int_t^\theta \int_{B_\rho}(1+u)^{\alpha-1}|\nabla u|^2 \xi^\lambda \leqq C.$$

Next recall the Gagliardo-Nirenberg estimate: let $m \geq 1, \gamma \in [1,+\infty)$ and $\nu \in [0,1]$ such that

$$\frac{1}{\gamma} = \nu(\frac{1}{2} - \frac{1}{N}) + \frac{1-\nu}{m}; \tag{4.5}$$

then there exists $C = C(N,m,\nu,\rho) > 0$ such that for any $w \in W^{1,2}(B_{\bar{\rho}}) \cap L^m(B_{\bar{\rho}})$,

$$\|w - \overline{w}\|_{L^\gamma(B_{\bar{\rho}})} \leq C\, \||\nabla w|\|^\nu_{L^2(B_{\bar{\rho}})} \|w - \overline{w}\|^{1-\nu}_{L^m(U)}. \tag{4.6}$$

We apply it to $w(x,t) = (1+u(x,t))^\beta$, and

$$\beta = \frac{1+\alpha}{2}, \qquad \gamma = 2 + \frac{2}{N\beta}, \qquad \nu = \frac{2}{\gamma}, \qquad m = \frac{1}{\beta}, \tag{4.7}$$

which satisfy (4.5). Therefore, for any $t \in (0,\theta)$,

$$\begin{aligned} \int_{B_{\bar{\rho}}}\left|(1+u(.,t))^\beta - \overline{w}(t)\right|^\gamma \leq C &\left(\int_{B_{\bar{\rho}}}(1+u(.,t))^{\alpha-1}|\nabla u(.,t)|^2\right) \times \\ &\left(\int_{B_{\bar{\rho}}}\left|(1+u(.,t))^\beta - \overline{w}(t)\right|^{1/\beta}\right)^{(1-\nu)\gamma\beta}. \end{aligned}$$

Now $\|\overline{w}(.)\|_{L^\infty((0,\theta))} \leq C$ because $\beta \in (0,1)$ and $\int_{B_\rho} u(.,t)$ is bounded; in turn we get

$$\int_{B_{\bar{\rho}}}\left|(1+u(x,t))^\beta - \overline{w}(t)\right|^{1/\beta} dx \leq C,$$

Therefore,

$$\int_{B_{\bar{\rho}}} (1+u(x,t))^{\beta\gamma}\, dx \leq C \int_{B_{\bar{\rho}}} (1+u(.,t))^{\alpha-1} \left|\nabla u(.,t)\right|^2 dx + C.$$

Integrating on $(0,\theta)$ we obtain

$$\int_0^\theta \int_{B_{\bar{\rho}}} (1+u(t))^{\beta\gamma}\, dx < C.$$

Observing that $\beta\gamma = \alpha + 1 + 2/N$, and α is defined by (4.3) we conclude to (4.2). ■

In order of proving Theorem 1.2 we show the following dichotomy property:

Proposition 4.3 *Assume $p, q > 1$. Let (u, v) be a positive solution of the system in $\Omega \times (0,T)$. Let $x_0 \in \Omega$. Then the following alternative holds:*

(i) Either there exists a ball $B(x_0,\rho) \subset \Omega$ such that $\int_0^T \int_{B(x_0,\rho)} (u^q + v^p) < \infty$ and two Radon measures $m_{1,\rho}$ and $m_{2,\rho}$ on $B(x_0,\rho)$, such that for any $\psi \in C_c^0(B(x_0,\rho))$,

$$\lim_{t\to 0} \int_{B(x_0,\rho)} u(.,t)\psi = \int_{B(x_0,\rho)} \psi dm_{1,\rho}, \qquad \lim_{t\to 0} \int_{B(x_0,\rho)} v(.,t)\psi = \int_{B(x_0,\rho)} \psi dm_{2,\rho}, \tag{4.8}$$

(ii) Or for any ball $B(x_0,\rho) \subset \Omega$ there holds $\int_0^T \int_{B(x_0,\rho)} (u^q + v^p) = \infty$ and then

$$\lim_{t\to 0} \int_{B(x_0,\rho)} (u(.,t) + v(.,t)) = \infty. \tag{4.9}$$

Proof. (i) Assume that there exists a ball $B(x_0,\rho) \subset \Omega$ such that $\int_0^T \int_{B(x_0,\rho)} (u^q+v^p) < \infty$. Then (4.8) follows from Lemma 4.1.

(ii) Suppose that for any ball $\int_0^T \int_{B(x_0,\rho)} (u^q + v^p) = \infty$. Consider a fixed $\rho > 0$ such that $B(x_0,\rho)$. We can assume $x_0 = 0$. We choose the test function ξ^λ, where ξ is defined at (3.3) and $\lambda > 2\max(p', q')$. Then

$$\frac{d}{dt}\left(\int_{B_\rho} u\xi^\lambda\right) + \int_{B_\rho} v^p \xi^\lambda = \int_{B_\rho} u(\Delta\xi^\lambda).$$

As above from (3.7), since λ is large enough,

$$\int_{B_\rho} u \left|\Delta\xi^\lambda\right| \leqq C(\int_{B_\rho} u^q \xi^\lambda)^{1/q},$$

where C depends on ρ. Let $0 < t < \theta < T$. Consider X, Y, Z, W defined by (4.1). Then we find with new constants $C > 0$

$$X_t(t) + W(t) \leqq C Z^{1/q}(t) \leqq \frac{Z(t)}{2} + C,$$
$$Y_t(t) + Z(t) \leqq C W^{1/p}(t) \leqq \frac{W(t)}{2} + C.$$

By addition

$$(X+Y)_t(t) + \frac{Z+W}{2}(t) \leqq C$$

By hypothesis $Z+W \notin L^1((0,T))$, then

$$\lim_{t\to 0}(X(t)+Y(t)) = \lim_{t\to 0}\int_{B_\rho}(u(.,t)+v(.,t))\xi^\lambda = \infty,$$

thus

$$\lim_{t\to 0}\int_{B_{\bar\rho}}(u(.,t)+v(.,t)) = \infty \tag{4.10}$$

for any $\bar\rho < \rho$, and the conclusion follows, since ρ is arbitrary. ■

As a direct consequence we deduce Theorem 1.2.

Proof of Theorem 1.2. Let

$$\mathcal{R} = \left\{ x_0 \in \Omega : \exists \rho > 0, \quad B(x_0,\rho) \subset \Omega, \quad \limsup \int_{B(x_0,\rho)} (u(.,t)+v(.,t) < \infty \right\},$$

and $\mathcal{S} = \Omega \backslash \mathcal{R}$. Then $\mathcal{R}$ is open, and from proposition 4.3, there exists unique Radon measures μ_1, μ_2 on $\mathcal{R}$ such that (1.7) holds, and (4.9) implies (1.8) on any open set $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{S} \neq \emptyset$. ■

Next we give more information when p,q are subcritical.

Proposition 4.4 *Assume $0 < p,q < 1+2/N$. Let (u,v) be a positive solution of the system in $\Omega\times(0,T)$. Let $x_0 \in \Omega$. Then then the eventuality (ii) of Theorem 4.3 is equivalent to:*

(iii) for any ball $B(x_0,\rho) \subset \Omega$ there holds

$$\int_0^T \int_{B(x_0,\rho)} u^q = \infty \text{ and } \int_0^T \int_{B(x_0,\rho)} v^p = \infty. \tag{4.11}$$

Proof. It is clear that (iii) implies (ii). Suppose that (iii) does not hold, and reduce to $x_0 = 0$. Then there exists a ball B_ρ such that for example

$$\int_0^T \int_{B_\rho} v^p < \infty.$$

Then for any $\bar\rho < \rho$ $\int_{B_{\bar\rho}} u(.,t)$ is bounded as $t \to 0$, from Lemma (4.1). Since $q < 1+2/N$, we obtain

$$\int_t^\theta \int_{B_{\rho'}} u^q dx \leqq C,$$

for any $\rho' < \bar\rho$, from Lemma 4.2. Then (ii) does not hold. ■

Remark 4.5 *Under the assumption (ii) or (iii) of Proposition 4.4, for any ball* $B_\rho = B(x_0,\rho) \subset \Omega$, $\int_{B_\rho} u(.,t)$ *and* $\int_{B_\rho} v(.,t)$ *are unbounded near* 0, *from Lemma 4.2. But we cannot prove that* $\lim_{t\to 0} \int_{B(x_0,\rho)} u(.,t) = \infty$ *or* $\lim_{t\to 0} \int_{B(x_0,\rho)} v(.,t) = \infty$, *even in the case* $p,q > 1$ *where (4.9) holds.*

We give a last result concerning the case where the two equations are sublinear.

Proposition 4.6 *Assume* $0 < p,q \leqq 1$. *Let* (u,v) *be a positive solution of the system in* $\Omega \times (0,T)$. *Then there exist two nonnegative Radon measures* μ_1, μ_2 *on* Ω, *such that for any* $\psi \in C_c^0(\Omega)$

$$\lim_{t\to 0} \int_\Omega u(.,t)\psi = \int_\Omega \psi d\mu_1, \qquad \lim_{t\to 0} \int_\Omega v(.,t)\psi = \int_\Omega \psi d\mu_2,$$

Proof. Consider any ball $B(x_0,\rho) \subset \Omega$, and assume $x_0 = 0$. Consider again X, Y, Z, W defined by (4.1). Here we find

$$W(t) = \int_{B_\rho} v^p(.,t)\xi^\lambda \leqq \int_{B_\rho} (v(.,t)+1)\xi^\lambda \leqq Y(t) + C,$$

$$Z(t) = \int_{B_\rho} u^q(.,t)\xi^\lambda \leqq \int_{B_\rho} (u(.,t)+1)\xi^\lambda \leqq X(t) + C,$$

and

$$\frac{d}{dt}(e^{\lambda\lambda_{1,\rho}t}X(t)) + e^{\lambda\lambda_{1,\rho}t}W(t) \geqq 0, \qquad \frac{d}{dt}(e^{\lambda\lambda_{1,\rho}t}Y(t)) + e^{\lambda\lambda_{1,\rho}t}Z(t) \geqq 0,$$

then the function $\Phi = e^{\lambda\lambda_{1,\rho}t}(X(t) + Y(t))$ satisfies $\Phi'(t) + \Phi(t) + Ce^{\lambda\lambda_{1,\rho}t} \geqq 0$, that is $(e^t(\Phi(t) + C(1+\lambda\lambda_{1,\rho})^{-1}e^{\lambda\lambda_{1,\rho}t})' \geqq 0$. Then $\Phi(t)$ has a limit as $t \longrightarrow 0$. ■

Open problems:

1) Can we extend Theorem 1.2 to the case $pq > 1$?
2) Can we extend Proposition 4.6 to the case $pq < 1$?

5 Removability results

Here we prove the removability of punctual singularities when p and q are supercritical.

Proof of Theorem 1.4. We can assume that $q \geq p \geq 1 + 2/N$. Let ω be a regular domain such that $\omega \subset\subset \Omega \backslash \{0\}$ and let $T_1 < T$. Then from (1.9) $u, v \in L^\infty(0,T_1; L^1(\omega))$; then from Lemma 4.2, $u \in L^q(\omega \times (0,T_1))$ and $v \in L^p(\omega \times (0,T_1))$. As in [6, Theorem 2], step 3, the functions defined on $\omega \times (-T,T)$ by

$$(\tilde{u},\tilde{v})(x,t) = \begin{cases} (u,v)(x,t) & \text{if } t > 0, \\ 0 & \text{if } t < 0, \end{cases}$$

satisfy $\tilde{u} \in L^q_{loc}(\omega \times (0,T))$, $\tilde{v} \in L^p_{loc}(\omega \times (0,T))$, and

$$\tilde{u}_t - \Delta\tilde{u} + \tilde{v}^p = 0, \qquad \tilde{v}_t - \Delta\tilde{v} + \tilde{u}^q = 0, \qquad \text{in } \mathcal{D}'(\omega \times (-T,T)). \tag{5.1}$$

It follows that $u, v \in C^{2,1}(\omega \times [0,T))$ and

$$u(x,0) = v(x,0) = 0, \qquad \forall x \in \omega.$$

Since $p < q$, we have $u^p \leqq u^q + 1$ from the Young inequality, thus the function

$$g = 2^{(1-p)/p}(u+v)$$

satisfies $g \in L^p(\omega \times (0,T_1))$, $g(x;0) = 0$ on $\omega \backslash \{0\}$ and

$$g_t - \Delta g + g^p \leqq 1$$

in $\omega \times (0,T)$. Following [6, Theorem 2], step 4, we deduce the key point estimate: there exists $C = C(N,p)$ and $\rho > 0$ such that $B(0,2\rho) \subset \Omega$, and $T_1 < T$ such that

$$g(x,t) \leqq \frac{C}{(t+|x|^2)^{1/(p-1)}} + C, \qquad \forall (x,t) \in B(0,\rho) \times (0,T_1)\,. \tag{5.2}$$

Since $p \geq 1 + 2/N$, it implies that $g \in L^1(B(0,\rho) \times (0,T_1))$. From [6, Theorem 2], step 5, it follows that $g \in L^p(B(0,\rho) \times (0,T_1))$, thus also u and v. We claim that a better estimate holds, adapted to the system:

$$\int_0^{T_1} \int_{B(0,\rho)} v^p < \infty \quad \text{ and } \int_0^{T_1} \int_{B(0,\rho)} u^q < \infty. \tag{5.3}$$

Indeed, consider a function $\zeta \in \mathcal{D}(\Omega \times (-T,T))$ with values in $[0,1]$, such that $\zeta = 1$ on $B(0,\rho) \times (0,T_1)$, and a function $\chi \in C^\infty(\mathbb{R})$, nondecreasing, with $\chi(t) = 0$ for $t \leqq 1$, $\chi(t) = 1$ for $t \geqq 2$; let $\chi_k(t) = \chi(kt)$ for any $k > 1$. Setting

$$D_k = \left\{(x,t) : 1/k < |x|^2 + t < 2/k\right\},$$

and using the test function

$$\varphi_k(x,t) = \chi_k(|x|^2 + t)\zeta(x,t),$$

we obtain

$$\int_0^T \int_{B_\rho} u^q \varphi_k = \int_0^T \int_{B_\rho} v(\varphi_k)_t + \int_0^T \int_{B_\rho} v\Delta\varphi_k,$$
$$\int_0^T \int_{B_\rho} v^p \varphi_k = \int_0^T \int_{B_\rho} u(\varphi_k)_t + \int_0^T \int_{B_\rho} u\Delta\varphi_k.$$

Consequently

$$\int_0^T \int_{B_\rho} u^q \varphi_k \leqq Ck \iint_{D_k} v + C, \qquad \int_0^T \int_{B_\rho} v^p \varphi_k \leqq Ck \iint_{D_k} u + C. \tag{5.4}$$

Next from (5.2), we have

$$\iint_{D_k} (u+v) \leqq \iint_{D_k} \left(\frac{C}{(t+|x|^2)^{1/(p-1)}} + C\right) \leqq \frac{C}{k} \tag{5.5}$$

Hence (5.3) follows from (5.4), (5.5) and the Fatou Lemma. As a consequence of (5.3), $\tilde{u} \in L^q_{loc}(\Omega \times (-T,T))$ and $\tilde{v} \in L^p_{loc}(\Omega \times (-T,T))$.

Following [6, Theorem 2], step 6, we have

$$\iint_{D_k} (u+v)\left(|(\chi_k)_t\zeta| + |(\Delta\chi_k)\zeta| + |\nabla\chi_k|\,|\nabla\zeta|\right) \leqq Ck\iint_{D_k}(u+v),$$

and from the Hölder inequality

$$k\iint_{D_k}(u+v) \leqq k\left(\iint_{D_k}(u+v)^p\right)^{1/p}|D_k|^{1/p'} \leqq C\left(\iint_{D_k}(u+v)^p\right)^{1/p}. \tag{5.6}$$

Since the right hand side of (5.6) tends to 0 from (5.3), we can pass to the limit as $k \to \infty$ in (**??**), and obtain

$$\int_0^T\int_{B_\rho} u^q\zeta = \int_0^T\int_{B_\rho} v\zeta_t + \int_0^T\int_{B_\rho} v\Delta\zeta, \qquad \int_0^T\int_{B_\rho} v^p\zeta = \int_0^T\int_{B_\rho} u\zeta_t + \int_0^T\int_{B_\rho} u\Delta\zeta,$$

and then

$$\tilde{u}_t - \Delta\tilde{u} + \tilde{v}^p = 0, \qquad \tilde{v}_t - \Delta\tilde{v} + \tilde{u}^q = 0, \qquad \text{in } \mathcal{D}'(\Omega \times (-T,T)).$$

Therefore $\tilde{u}, \tilde{v} \in C^{2,1}(\Omega \times (-T,T))$, and $u(x,0) = v(x,0) = 0$ on Ω. ∎

Open problem: In the elliptic problem (1.4) in $B(0,1)\backslash\{0\}$, it was shown in [4, Corollary 1.2] that the singularities at 0 are removable as soon as

$$\max(2a, 2b) \leqq N - 2.$$

In the case of system (1.1), an open question is to know if the initial punctual singularities at 0 are removable whenever

$$\max(a,b) \leqq \frac{N}{2},$$

a condition which is obviously satisfied when $p, q \geqq 1 + 2/N$.

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